Brownian Motion and Harmonic Functions

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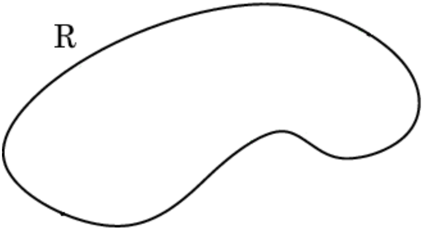
2012.08.02

Laplace's Equation



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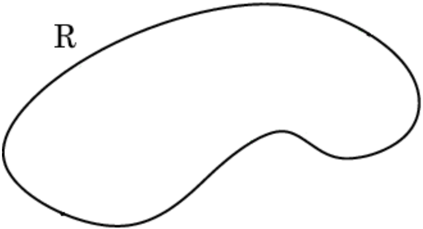

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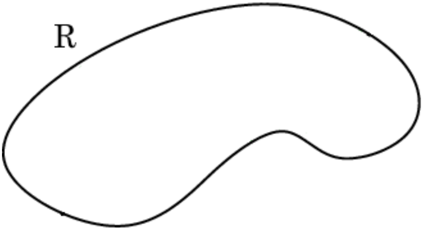

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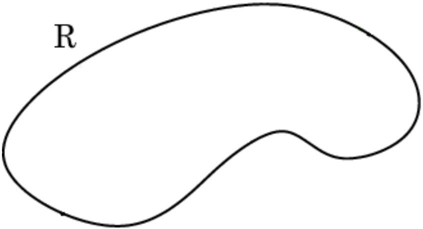

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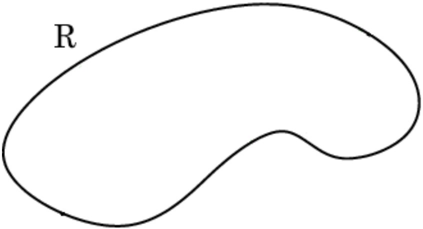

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- After a sufficient amount of time, the temperature at points interior are time-independent
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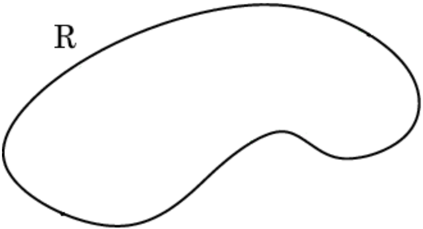
  - In nice regions, the solution is well-known



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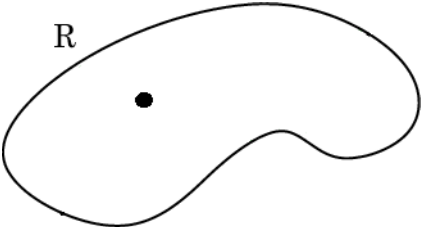

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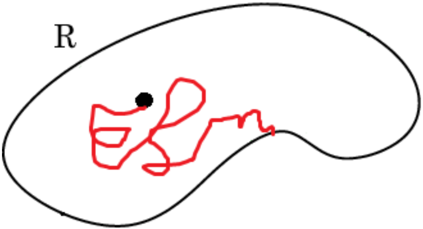

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- We can use random walks to simulate Brownian motion
  - Specifically, the random walks on circles (RWoC) and spheres (RWoS)
  - We simulated Brownian motion in various regions and studied the probability density functions (PDFs) of the point of first encounter in these regions

Walk on Circles

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- Continue until process until you hit boundary of area

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  - Upper Half-Space
  - Sphere

Simulation: Half Plane

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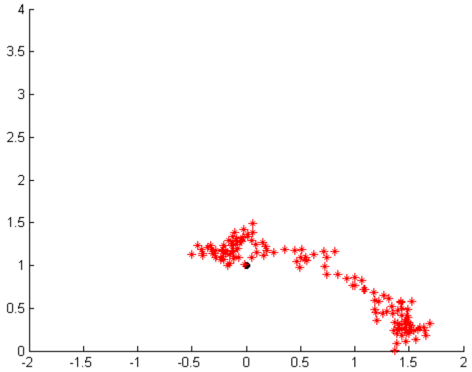

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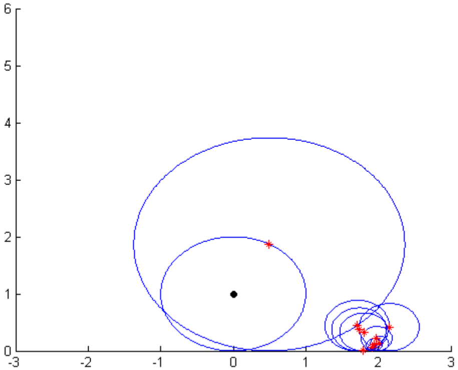
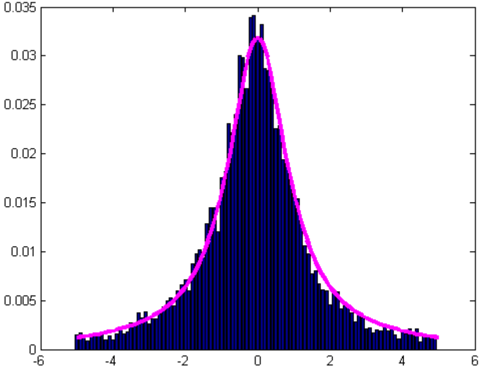


- How did our simulation perform?



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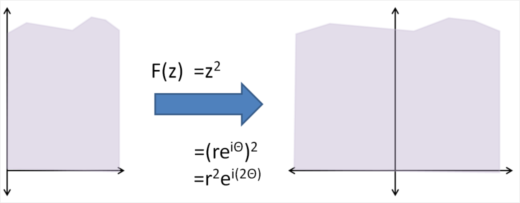
  - Riemann Mapping Theorem



- PDF
  - x-axis

$$\frac{1}{\pi}\frac{4x_0y_0\tau}{(x_0{}^2-y_0{}^2-\tau^2)^2+(2x_0y_0)^2}$$

  - y-axis

$$\frac{1}{\pi}\frac{4x_0y_0\tau}{(x_0{}^2-y_0{}^2+\tau^2)^2+(2x_0y_0)^2}$$

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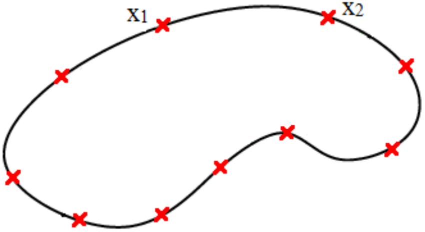

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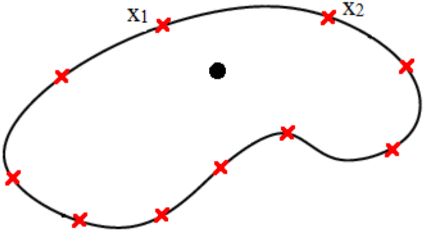

Real World Applications

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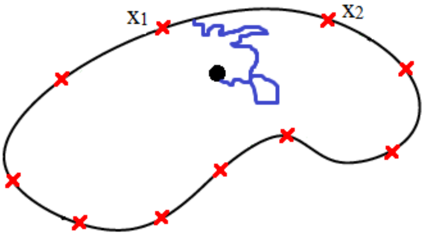

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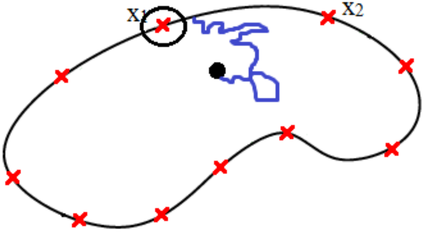

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- But we can do better, u_0 can be up to a $2(10) - 1$ degree polynomial

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  - Pick x_i as the roots of $p_n(x)$

Checking the Efficiency

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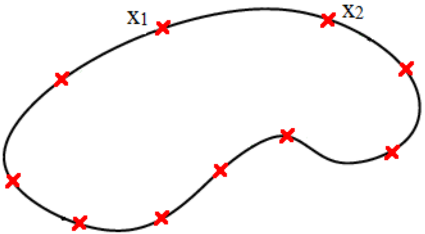
&= \sum_{i=1}^{10} u_0(x_i)D_i
\end{aligned}$$

What Have We Done?



- So if u_0 is a “nice” (smooth) function, then

$$
\begin{aligned}
u(x,y) &= \int_{-\infty}^{\infty} D(\tau)u_0(\tau)d\tau \\
&\approx \sum_{i=1}^{10} u_0(x_i)D_i
\end{aligned}
$$

- This will be a *good* approximation

Summary

- Brownian Motion and Laplace's Equation
- Walk on Circles and Spheres
- Simulating Walk on Circles and Spheres in different regions
- Probability Density Functions and Conformal Mapping Techniques
- Less "Expensive" Real World Applications

- NSA
  - Grant: H98230-11-10222.
- Dr. Igor Nazarov
- Dr. Nick Boros

Thanks for Listening!

Questions?